Math 2263

Quiz 12

Name Section Score

(8 points) Let $F(x,y) = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$,

(a) Show that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

(b) Show that $\int\limits_C \mathbf{F}\cdot\boldsymbol{dr}$ is NOT independent of path. Does this contradict Theorem 6? [Hint: Compute $\int\limits_{C_1} \mathbf{F}\cdot\boldsymbol{dr}$ and $\int\limits_{C_2} \mathbf{F}\cdot\boldsymbol{dr}$, where $C_1$ and $C_2$ are the upper and lower halves of the circle $x^2+y^2=1$ from (1, 0) to (-1, 0)].

Theorem 6: Let $F = P\mathbf{i}+Q\mathbf{j}$ be a vector field on an open simply-connected region $D$. Suppose $P$ and $Q$ have continuous first-order derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D, then $F$ is conservative.

Solution: (a) $P = -\frac{y}{x^2+y^2}$, $\frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$, and $Q = \frac{x}{x^2+y^2}$, $\frac{\partial Q}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$. Thus, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

(b) $C_1 : x = \cos t, y = \sin t, 0 \le t \le \pi$; $C_2$: $x = \cos t, y = \sin t$, $t = 2\pi$ to $t = \pi$. Then

$$\int_{C_1} \mathbf{F}\cdot\boldsymbol{dr} = \int_0^{\pi} \frac{(-\sin t)(-\sin t)+(\cos t)(\cos t)}{\cos^2 t+\sin^2 t}dt = \int_0^{\pi} dt = \pi,$$

and

$$\int_{C_2} \mathbf{F}\cdot\boldsymbol{dr} = \int_{2\pi}^{\pi} dt = -\pi.$$

Since these aren't equal, the line integral of $\mathbf{F}$ isn't independent of path. This doesn't contradict Theorem 6, since the domain of $\mathbf{F}$, which is $\mathbb{R}^2$ except the origin, isn't simply-connected.

(7 points) Let $D$ be a region bounded by a simple closed path $C$ in the $xy$-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x},\bar{y})$ of $D$ are

$$\bar{x} = \frac{1}{2A}\oint\limits_C x^2dy,\ \bar{y} = -\frac{1}{2A}\oint\limits_C y^2dx$$

where $A$ is the area of $D$.

Solution: By Green's theorem, $\frac{1}{2A}\oint\limits_C x^2dy = \frac{1}{2A}\iint\limits_D 2xdA = \frac{1}{A}\iint\limits_D xdA = \bar{x}$, and $-\frac{1}{2A}\oint\limits_C y^2dx = -\frac{1}{2A}\iint\limits_D -2ydA = \frac{1}{A}\iint\limits_D ydA = \bar{y}$.